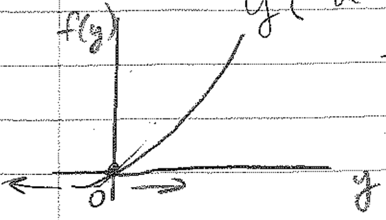
2.5 Autonomous Equations and Population Dynamics

p. 88: 1, 3, 7, 8, 12, 23

1.) $\frac{dy}{dt} = ay + by^2$; $a > 0$, $b > 0$, $y_0 \geq 0$

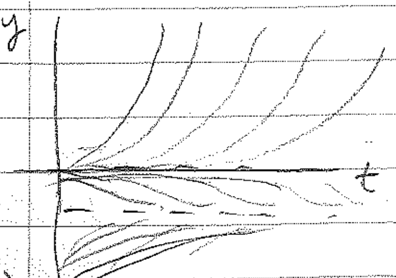
$y(a+by) = 0$ when $y = 0$ or $y = -\frac{a}{b} < 0$



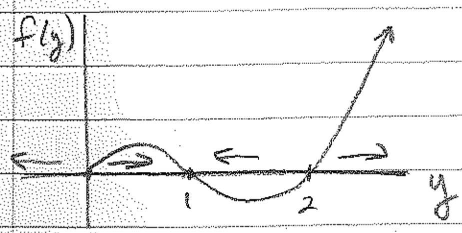
for $y_0 \geq 0$, the only equilibrium point is $y = 0$

$f'(y) = a + 2by \rightarrow f'(0) = a > 0$

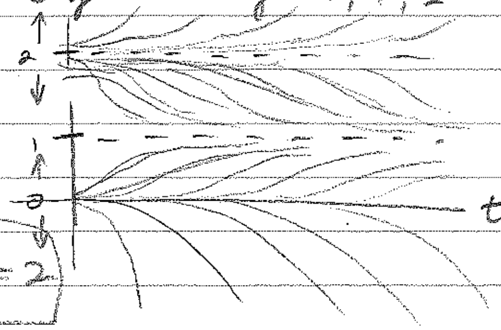
and so $\phi(t) = 0$ is $\boxed{\text{unstable}}$



3.) $\frac{dy}{dt} = y(y-1)(y-2)$; $y_0 \geq 0$



$\frac{dy}{dt} = 0$ when $y = 0, 1, 2$

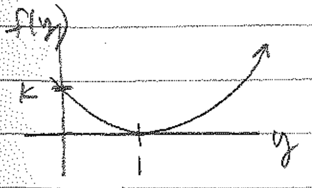


stable @ $y = 1$

unstable @ $y = 0$ & $y = 2$

7.) a.) $\frac{dy}{dt} = k(1-y)^2 = 0$ when $y = 1$, so $\phi(t) = 1$

b.)



c.) $(1-y)^{-2}\,dy = k\,dt$

$\frac{1}{1-y} = kt + c \rightarrow c = \frac{1}{1-y_0}$

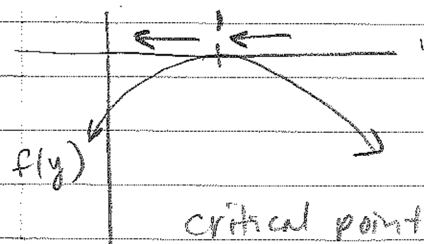
$1 - y = \frac{1}{kt+c}$

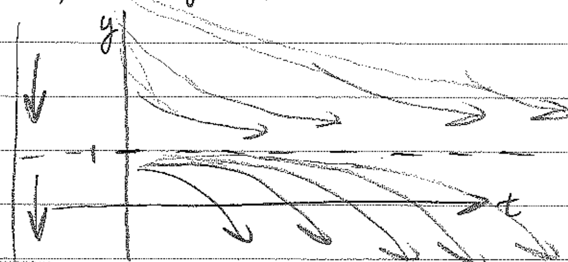
$y = 1 - \frac{1}{kt + \frac{1}{1-y_0}} = 1 - \frac{1-y_0}{kt(1-y_0)+1} \rightarrow \boxed{y = \frac{kt(1-y_0)+y_0}{kt(1-y_0)+1}}$

In part (b) $\frac{dy}{dt} = f(y)$ always (+)

so y is always increasing

8.) $\frac{dy}{dt} = -k(y-1)^2$; $k>0$, $-\infty < y_0 < \infty$



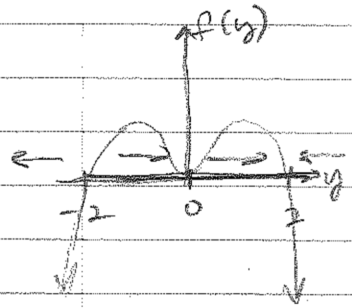


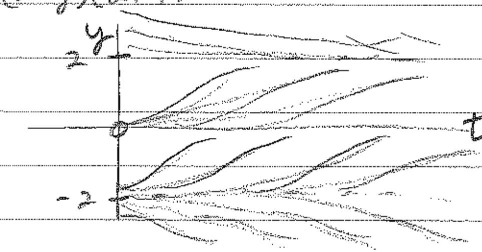
critical point $y=1$

$\boxed{y = 1 \text{ is semi-stable}}$

12.) $\frac{dy}{dt} = y^2(4-y^2)$; $-\infty < y_0 < \infty$

$y^2(2-y)(2+y)$





$\boxed{\begin{array}{l} \text{stable @ } y=2 \\ \text{semistable @ } y=0 \\ \text{unstable @ } y=-2 \end{array}}$

23.) $\frac{dy}{dt} = -\beta y$ $\qquad$ $\frac{dx}{dt} = -\alpha x y$ $\qquad$ x = susceptibles
y = carriers

a.) $\frac{1}{y}\,dy = -\beta\,dt$
$\ln|y| = -\beta t + C$
$y = y_0 e^{-\beta t}$

b.) $\frac{dx}{dt} = -\alpha x y_0 e^{-\beta t}$

$\frac{1}{x}\,dx = -\alpha y_0 e^{-\beta t}\,dt$

$\ln|x| = \frac{\alpha}{\beta} y_0 e^{-\beta t} + C$

$X = X_0 e^{\frac{\alpha y_0}{\beta} e^{-\beta t}}$ or $X = x_0 e^{\frac{\alpha y_0}{\beta}(1-e^{-\beta t})}$

c.) As $t \to \infty$ $\quad y \to 0$
and $\quad x \to \boxed{x_0 e^{\frac{\alpha}{\beta} y_0}}$

As time passes, the carriers disappear, so the proportion of the population that escapes is equal to the proportion of susceptibles left.

$\ln|x| = \frac{\alpha}{\beta} y_0 \left(1 - e^{-\beta t}\right)$

$\frac{1}{x}\,dx = \frac{-\alpha}{\beta} y_0 (-\beta)\left(-e^{-\beta t}\right) = -\alpha y_0 e^{-\beta t}$

2.6 Exact Equations and Integrating Factors

p. 99 : 2, 3, 10, 14, 22, 26

2.) $(2x+4y)+(2x-2y)y'=0$ $\quad M_y \neq N_x$ $\boxed{\text{not exact}}$

$M_y = 4 \qquad N_x = 2$

3.) $(3x^2-2xy+2)\,dx+(6y^2-x^2+3)\,dy=0$ $\quad\boxed{\text{exact}}$

$M_y = -2x \qquad N_x = -2x \qquad M_y = N_x$

$\psi_x = 3x^2-2xy+2 \qquad \psi_y = 6y^2-x^2+3$

$\psi = x^3-\underline{x^2y}+2x+g(y) \qquad \psi = 2y^3-\underline{x^2y}+3y+h(x)$

$$\psi = \boxed{x^3+2x-x^2y+2y^3+3y=C}$$

10.) $(y/x+6x)\,dx+(\ln x-2)\,dy=0$; $x>0$

$M_y = 1/x \qquad N_x = 1/x \qquad M_y = N_x$ $\boxed{\text{exact}}$

$\psi_x = y/x+6x \qquad \psi_y = \ln x - 2$

$\psi = y\underline{\ln x}+3x^2+g(y) \qquad \psi = y\underline{\ln x}-2y+h(x)$

$$\psi = \boxed{y\ln x+3x^2-2y=C}$$

14.) $(9x^2+y-1)\,dx+(x-4y)\,dy=0$; $y(1)=0$

$M_y = 1 \qquad N_x = 1$ $\boxed{\text{exact}}$

$\psi = 3x^3+\underline{xy}-x+g(y) \qquad \psi = \underline{xy}-2y^2+h(x)$

$\psi = 3x^3-x+xy-2y^2=C$

$y(1)=0 \rightarrow 3-1=C \rightarrow C=2$

$\boxed{3x^3-x+xy-2y^2=2}$ implicit

$2y^2-xy-3x^3+x+2$

$2y^2-xy-(3x^3-x-2)=0$

$$y = \frac{x \pm \sqrt{x^2+4(2)(3x^3-x-2)}}{4} \qquad x^2+8(3x^3-x-2)>0$$

$\boxed{x > 0.985}$

22.) $(x+2)\sin y\, dx + x\cos y\, dy = 0 \quad \mu(x,y) = xe^x$

$M_y = (x+2)\cos y \neq N_x = \cos y \rightarrow$ not exact

$xe^x(x+2)\sin y\, dx + x^2e^x \cos y\, dy = 0$

$M_y = xe^x(x+2)\cos y$ ✓ $\quad N_x = \cos y(2xe^x + x^2e^x)$

$= xe^x(2+x)\cos y$ ✓

x^2	$+\ e^x$
$2x$	$-\ e^x$
2	$+\ e^x$
0	e^x

$\Psi_x = (x^2e^x + 2xe^x)\sin y \qquad \Psi_y = x^2e^x\cos y$

$\left\{\begin{array}{l}\Psi = (x^2e^x - 2xe^x + \cancel{2}e^x + 2\cancel{x}e^x - \cancel{2}e^x)\sin y + g(y) \\ \Psi = x^2e^x \sin y + g(y)\end{array}\right.$

$\left.\begin{array}{l}\Psi = x^2e^x\sin y + g(y) \\ \Psi = x^2e^x \sin y + h(x)\end{array}\right\} \rightarrow \quad \Psi = \boxed{x^2e^x\sin y = C}$

26.) $y' = e^{2x} + y - 1 \qquad (e^{2x} + y - 1)\, dx - 1\, dy = 0$

$\frac{M_y - N_x}{N} = \frac{1-0}{-1} = -1 \qquad \frac{d\mu}{dx} = -\mu \qquad \frac{1}{\mu}\, d\mu = -1\, dx$

$\ln\mu = -x + c$

$\mu = e^{-x}$

$e^{-x}(e^{2x} + y - 1)\, dx - e^{-x}\, dy = 0$

$(e^x + e^{-x}y - e^{-x})\, dx - e^{-x}\, dy = 0$

$\Psi = e^x - e^{-x}y + e^{-x} + g(y) \qquad \Psi = -e^{-x}y + h(x)$

$\boxed{e^x - e^{-x}y + e^{-x} = C} \qquad +e^{-x}y = C + e^x + e^{-x}$

$y = Ce^x + e^{2x} + 1$

or

$\boxed{y = Ce^x + e^{2x} + 1}$

2.7 Numerical Approximation: Euler's Method

p. 108: 3, 5, 6, 8

3.) $y' = 0.5 - t + 2y$, $y(0) = 1$

a.) $h = 0.1$

t	y	y'
0	1	2.5
0.1	1.25	2.9
0.2	1.54	3.38
0.3	1.878	3.956
0.4	2.2736	

b.) $h = 0.05$

t	y	y'
0	1	2.5
0.05	1.125	2.7
0.1	1.26	2.92
0.15	1.406	3.162
0.2	1.5641	3.4282
0.25	1.73551	3.72102
0.3	1.92156	4.04312
0.35	2.1237	4.3974
0.4	2.3435	

c.) $h = 0.025$

t	y
0	1
0.1	1.26551
0.2	1.57746
0.3	1.94586
0.4	2.38287

$y' - 2y = 0.5 - t$

$e^{-2t}y' - 2e^{-2t}y = .5e^{-2t} - te^{-2t}$

$e^{-2t}y = \int .5e^{-2t} - te^{-2t}$

t	$+\ e^{-2t}$
1	$-\ -\frac{1}{2}e^{-2t}$
0	$\frac{1}{4}e^{-2t}$

$e^{-2t}y = -\frac{1}{4}\cancel{e^{-2t}} + \left(+\frac{1}{2}te^{-2t} + \frac{1}{4}\cancel{e^{-2t}}\right) + C$

$y = \frac{1}{2}t + Ce^{2t} \rightarrow C = 1 \rightarrow \boxed{y = \frac{1}{2}t + e^{2t}}$

$y(0) = 1$

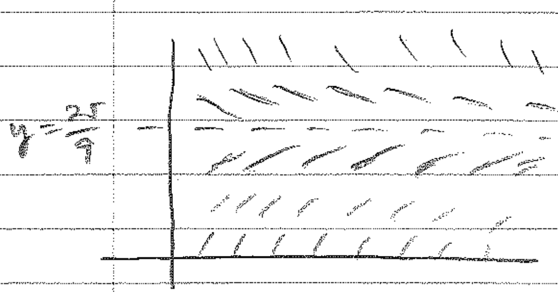
0	1
.1	1.2714
.2	1.59182
.3	1.97212
.4	2.42554

5.) $y' = 5 - 3\sqrt{y}$

$5 = 3\sqrt{y}$

$25 = 9y$

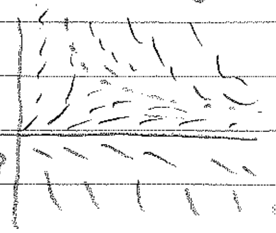
$y = 25/9$ equilibrium



converges toward $y = 25/9$

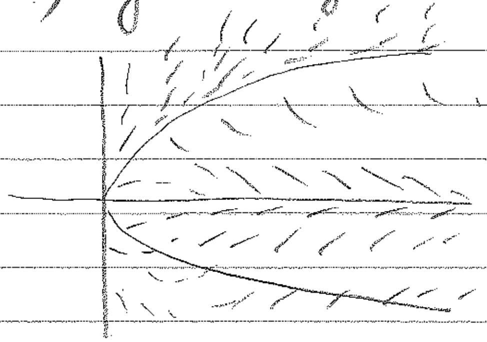
6.) $y' = y(3 - ty)$ $\qquad y = 0 \quad y = 3/t$

positive initial cond converges to $y = 3/t$
negative initial cond diverges



8.) $y' = -ty + 0.1y^3$ $\qquad y(0.1y^2 - t)$

$y = 0 \qquad y = \pm\sqrt{10t}$



converge toward $y = 0$ if init. cond "inside" $y = \pm\sqrt{10t}$

diverge from $y = \pm\sqrt{10t}$ if init. cond "outside"