2.1 Linear Equations, Integrating Factors

p. 39: 2, 15, 20, 26

2.) $y' - 2y = t^2 e^{2t}$ $\quad \mu = e^{-2t}$ $\quad \boxed{\text{as } t \to \infty,\ y \to \infty}$

$e^{-2t}\frac{dy}{dt} - 2e^{-2t}y = t^2 e^{-2t} \cdot e^{2t} = t^2$

$\frac{d}{dt}\left[e^{-2t}y\right] = t^2 \to e^{-2t}y = \frac{1}{3}t^3 + c$

$\boxed{y = e^{2t}\left(\frac{1}{3}t^3 + c\right)}$

15.) $ty' + 2y = t^2 - t + 1$ $\quad p(t) = \frac{2}{t}$ $\quad \mu(t) = e^{2\ln t} = t^2$

$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$ $\quad g(t) = t - 1 + \frac{1}{t}$ $\quad \boxed{y(1) = \frac{1}{2}}$

$t^2y' + 2ty = t^3 - t^2 + t \to \frac{d}{dt}\left[t^2 y\right] = t^3 - t^2 + t$

$t^2 y = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c \to y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2}$

$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \to \boxed{y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}}$

$c = \frac{1}{12}$

20.) $ty' + (t+1)y = t$ $\quad y(\ln 2) = 1$ $\quad p(t) = \frac{t+1}{t} = 1 + \frac{1}{t}$

$y' + \frac{t+1}{t}y = 1$ $\quad g(t) = 1$ $\quad \mu(t) = e^{t + \ln t} = te^t$

$(te^t)y' + (1 + \frac{1}{t})(te^t)y = te^t$

tabular (by parts)

t	$+$	e^t
1	$-$	e^t
0		e^t

$\frac{d}{dt}\left[te^t y\right] = te^t$

$te^t y = te^t - e^t + c$

$y = 1 - \frac{1}{t} + \frac{c}{te^t} \to 1 = 1 - \frac{1}{\ln 2} + \frac{c}{(\ln 2)(2)} \to c = 2$

$\boxed{y = 1 - \frac{1}{t} + \frac{2}{te^t}}$

26.) $(\sin t)y' + (\cos t)y = e^t$ $\quad y(1) = a$ $\quad$ a.) as $t \to 0$ $y \to \infty$ for $y > a_0$

$y' + (\cot t)y = \frac{e^t}{\sin t}$ $\quad p(t) = \cot t$ $\quad g(t) = \frac{e^t}{\sin t}$ $\quad y \to -\infty$ for $y < a_0$

$\mu(t) = e^{\ln \sin t} = \sin t$

Well, that just took me back to where I started.
(maybe I should check for product rule from the beginning!)

26 (cont.) →

26 (cont.) b.) $\frac{d}{dt}\left[\sin t \cdot y\right] = e^t \rightarrow y \sin t = e^t + c$

$y = \frac{e^t + c}{\sin t} \rightarrow a = \frac{e + c}{\sin(1)} \rightarrow a\sin(1) = e + c$

$c = a\sin(1) - e$

$\boxed{y = \frac{e^t + a\sin(1) - e}{\sin t}}$

$0 = e^0 + a\sin(1) - e$

$a\sin(1) = e - 1$

$\boxed{a = (e-1)/\sin(1)}$

c.) $y \rightarrow 1$ for $a = a_0$

2.2 Separable Equations

p.47: 1, 4, 5, 6, 10, 12, 18, 22, 25 (no need to find max)

1.) $\frac{dy}{dx} = x^2/y$

$y\,dy = x^2\,dx$

$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C_1$

$6\left(\frac{1}{2}y^2 - \frac{1}{3}x^3 = C_1\right)$

$\boxed{3y^2 - 2x^3 = C}$

4.) $\frac{dy}{dx} = \frac{3x^2-1}{3+2y}$

$(3+2y)\,dy = (3x^2-1)\,dx$

$3y + y^2 = x^3 - x + C$

$\boxed{3y + y^2 - x^3 + x = C}$

5.) $\frac{dy}{dx} = (\cos^2 x)(\cos^2 2y)$

$\int \sec^2(2y)\,dy = \int \cos^2 x\,dx$

$\frac{1}{2}\tan(2y) = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

$\boxed{2\tan(2y) - 2x - \sin(2x) = C}$

$\frac{1}{2}\int(\cos(2x)+1)\,dx$
$= \frac{1}{2}\cdot\frac{1}{2}\sin(2x) + \frac{1}{2}x$

6.) $x\frac{dy}{dx} = (1-y^2)^{1/2}$

$\int \frac{dy}{(1-y^2)^{1/2}} = \int \frac{dx}{x}$

$\sin^{-1} y = \ln|x| + C$

$\boxed{y = \sin(\ln|x| + C)}$

10.) $\frac{dy}{dx} = \frac{1-2x}{y}$; $y(1) = -2$

$y\,dy = (1-2x)\,dx$

$\frac{1}{2}y^2 = x - x^2 + C$

$\frac{1}{2}(4) = 1 - 1 + C \rightarrow C = 2$

$\frac{1}{2}y^2 = x - x^2 + 2$

$y^2 = 2x - 2x^2 + 4$ $\quad \boxed{-1 < x < 2}$

$\boxed{y = -\sqrt{2x - 2x^2 + 4}}$ $\quad y < 0$

12.) $\frac{dr}{d\theta} = r^2/\theta$; $r(1) = 2$

$r^{-2}\,dr = \frac{d\theta}{\theta}$

$-\frac{1}{r} = \ln|\theta| + C$

$-\frac{1}{2} = \ln(1) + C \rightarrow C = -\frac{1}{2}$

$r = \frac{-1}{\ln|\theta| - \frac{1}{2}}$

$\boxed{r = \frac{2}{1 - 2\ln|\theta|}}$

$0 < \theta < e^{1/2}$

18.) $\frac{dy}{dx} = \frac{e^{-x} - e^{x}}{3+4y}$; $y(0) = 1$

$(3+4y)\,dy = (e^{-x} - e^{x})\,dx$

$3y + 2y^2 = -e^{-x} - e^{x} + C$

$3 + 2 = -1 - 1 + C \rightarrow C = 7$

$2y^2 + 3y + (e^{-x} + e^{x} - 7) = 0$

$y = \frac{-3 + \sqrt{9 - 8(e^{-x} + e^{x} - 7)}}{4}$

$\boxed{y = -\frac{3}{4} + \frac{1}{4}\sqrt{65 - 8e^{-x} - 8e^{x}}}$

$-2.08 < x < 2.08$

22.) $\frac{dy}{dx} = \frac{3x^2}{3y^2-4}$; $y(1) = 0$

$(3y^2-4)\,dy = 3x^2\,dx$

$y^3 - 4y = x^3 + C$

$0 = 1 + C \rightarrow C = -1$

$\boxed{y^3 - 4y = x^3 - 1}$

$3y^2 - 4 = 0$

$y^2 = 4/3$

$y = \pm \frac{2}{\sqrt{3}}$

$\frac{8}{3\sqrt{3}} - \frac{8}{\sqrt{3}} = x^3 - 1$

$\frac{-16}{3\sqrt{3}} = x^3 - 1 \rightarrow |x^3 - 1| < \frac{16}{3\sqrt{3}}$

25.) $\frac{dy}{dx} = \frac{2\cos 2x}{3+2y}$; $y(0) = -1$

$(3+2y)\,dy = 2\cos 2x\,dx$

$3y + y^2 = \sin 2x + C$

$-3 + 1 = 0 + C \rightarrow C = -2$

$\boxed{3y + y^2 - \sin 2x = -2}$

explicit solution $y^2 + 3y + 2 - \sin 2x = 0$

$y = \frac{-3 \pm \sqrt{9 - 4(2 - \sin 2x)}}{2}$

$\boxed{y = -\frac{3}{2} + \frac{1}{2}\sqrt{1 + 4\sin 2x}}$

$1 + 4\sin 2x > 0$?

$\sin 2x = -1/4$.

2.3 Modeling with 1st Order Diffy Q's

p.61: 9, 13, 32

9.) $\frac{dS}{dt} = rS + k \quad S(0) = 8000 \quad r = 0.10$

$S(t) = ce^{rt} - (k/r) \quad c = S_0 + (k/r)$

$S(t) = 8000e^{t/10} - 10k(e^{t/10} - 1)$

$0 = 8000e^{3/10} - 10k(e^{3/10} - 1) \rightarrow \boxed{k = \$3086.64/\text{yr.}}$

$8000e^{3/10} - 8000 = \$2798.87$ (book says $1289.92 ?)

13.) $2M = M_0 e^{7k} \quad k = \frac{\ln 2}{7} \quad M(t) = 200{,}000e^{\frac{\ln 2}{7}t}$

$M(t) = 200{,}000e^{\frac{\ln 2}{7}t} - 201977.31(e^{\frac{\ln 2}{7}t} - 1)$

or $\boxed{M(t) = 201977.31 - 1977.31e^{(\frac{\ln 2}{7})t}}$

32.) $(1 + y'^2)y = k^2$

a.) $y'^2 = \frac{k^2}{y} - 1 = \frac{k^2 - y}{y}$

$y'(x) = \sqrt{\frac{k^2 - y}{y}}$ positive root b/c y is an increasing function of x

b.) Let $y = k^2 \sin^2 t$

$dy = 2k^2 \sin t \cos t \, dt$

$\frac{dy}{dx} = \frac{2k^2 \sin t \cos t \, dt}{dx} = \sqrt{\frac{k^2 - k^2\sin^2 t}{k^2 \sin^2 t}} = \sqrt{\frac{\cos^2 t}{\sin^2 t}} = \frac{\cos t}{\sin t}$

$\frac{2k^2 \sin t \cancel{\cos t} \, dt}{dx} = \frac{\cancel{\cos t}}{\sin t} \quad dx = 2k^2 \sin^2 t \, dt$

c.) Letting $\theta = 2t$ $\quad \int k^2 \sin^2\left(\frac{\theta}{2}\right) d\theta = \int dx$

use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\quad \frac{k^2}{2}\int(1 - \cos\theta)\, dt = x$

$x = \frac{k^2(\theta - \sin\theta)}{2}$

$$dy = 2k^2 \sin t \cos t \, dt$$

$$y = \int k^2 \sin\tfrac{\theta}{2} \cos\tfrac{\theta}{2} \, d\theta = \qquad u = \sin\tfrac{\theta}{2}$$

$$du = \tfrac{1}{2}\cos\tfrac{\theta}{2}\, d\theta$$

$$= 2k^2 \int u \, du = 2k^2 \cdot \tfrac{1}{2}u^2 + c$$

$$k^2 \sin^2(\theta/2) = k^2 \frac{(1-\cos\theta)}{2}$$

$$y = \frac{k^2(1-\cos\theta)}{2}$$

d.) $1 = k^2(\theta - \sin\theta)/2 \qquad 2 = k^2(1-\cos\theta)/2$

$$k^2 = \frac{2}{\theta - \sin\theta} \qquad k^2 = \frac{4}{1-\cos\theta}$$

$$\frac{2}{\theta - \sin\theta} = \frac{4}{1-\cos\theta}$$

$$4\theta - 4\sin\theta = 2 - 2\cos\theta$$

$$\theta = 1.40137$$

$$k^2 = \frac{4}{1-\cos\theta} = 4.811 \rightarrow \boxed{k = 2.193}$$

2.4 Differences between Linear and Nonlinear Eqns

p.75: 2, 3, 6, 8, 14, 22, 27

2.) $t(t-4)y' + y = 0$; $y(2) = 1$

$\boxed{0 < t < 4}$

3.) $y' + (\tan t)y = \sin t$; $y(\pi) = 0$

$\boxed{\pi/2 < t < 3\pi/2}$

6.) $(\ln t)y' + y = \cot t$; $y(2) = 3$

$\cancel{0} < t < \pi \rightarrow \boxed{1 < t < \pi}$

no solution
when $t = 1$

8.) $y' = (1 - t^2 - y^2)^{1/2} \rightarrow f(t, y) = (1 - t^2 - y^2)^{1/2}$

$\frac{\partial f}{\partial y} = \frac{1}{2}(1 - t^2 - y^2)^{-1/2}(-2y) = \frac{-y}{\sqrt{1 - t^2 - y^2}}$

$-(t^2 + y^2) = -1$

$\boxed{t^2 + y^2 < 1}$

14.) $\frac{dy}{dt} = 2t + y^2$; $y(0) = y_0$

$y^{-2}dy = 2t\,dt$

$-\frac{1}{y} = t^2 + c$

$-\frac{1}{y_0} = c$

$-\frac{1}{y} = t^2 - \frac{1}{y_0} = \frac{t^2 y_0 - 1}{y_0}$

$y = \frac{-y_0}{t^2 y_0 - 1} = \frac{y_0}{1 - t^2 y_0}$

If $y_0 > 0$, solutions exist for $t^2 < \frac{1}{y_0}$

If $y_0 \leq 0$, solutions exist for all t

22.) $y' = \dfrac{-t + (t^2+4y)^{1/2}}{2}$; $y(2) = -1$

$y_1(t) = 1-t$ $\qquad$ $y_2(t) = -t^2/4$

$y_1' = -1$ $\qquad$ $y_2' = -t/2$

$-1 = \dfrac{-t + (t^2-4)^{1/2}}{2}$ $\qquad$ $-\dfrac{t}{2} = \dfrac{-t + (t^2-2y)^{1/2}}{2}$

$-1 = \dfrac{-2+0}{2} = -1$ ✓ $\qquad$ $-1 = \dfrac{-2+0}{2} = -1$ ✓

27.) $y' + p(t)y = q(t)y^n$

a.) For $n=0$ $\quad y' + p(t)y = q(t)$

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)\, q(s)\, ds + c\right] \ ; \ \mu(t) = e^{\int p(t)\,dt}$$

For $n=1$ $\quad y' + p(t)y = q(t)y$

$$y' + (p(t) - q(t))\,y = 0$$

$$\frac{y'}{y} = q(t) - p(t)$$

$$\ln|y| = \int q(t) - p(t)\, dt \rightarrow y = e^{\int q(t) - p(t)\, dt}$$

b.) Using $v = y^{1-n}$ and dividing the original by y^n

$$y^{-n}y' + p(t)y^{1-n} = q(t)$$
$$v'y' + p(t)v = q(t)$$

$$v' = (1-n)y^{-n}y' \rightarrow y^{-n}y' = \frac{v'}{n-1}$$

$$\frac{v'}{n-1} + p(t)v = q(t)$$