1.1 Some Basic Mathematical Models; Direction Fields

p.7: 3, 4, 7, 12, 15, 17, 18, 20, 21, 22,

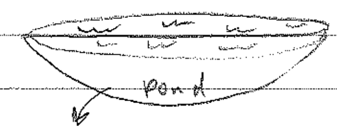
3.) $y' = 3 + 2y$ equilibrium @ $y = -3/2$
for $y > -3/2$ slopes are positive and increasing
For $y < -3/2$ slopes are negative and decreasing
y diverges from $-3/2$ as $t \to \infty$

4.) $y' = -1 - 2y$ equilibrium @ $y = -\frac{1}{2}$
$y \to -\frac{1}{2}$ as $t \to \infty$

7.) All solutions approach $y = 3$: $y' = 3 - y$

12.) $y' = -y(5-y)$
For $y < 0$ and $0 < y < 5$, solutions approach $y = 0$
For $y > 5$, solutions diverge from $y = 5$ as $t \to \infty$

15.) j 17.) g 18.) b 20.) e

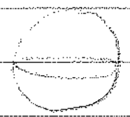
21.) 

300 gal./hr.

$\leftarrow 300 \frac{\text{gal}}{\text{hr}}$ with $.01 \frac{\text{gram}}{\text{gal.}}$

Let y be the amount of chemical in the 1,000,000 gal pond.

a.) Rate of change per hr. $\frac{dy}{dt} = 300\left(.01 - \frac{y}{1{,}000{,}000}\right)$
b.) 10,000 grams (independent of how much present initially)

22.) 

$\frac{dV}{dt} = -k(4\pi r^2)$ $\quad V = \frac{4}{3}\pi r^3$

$r = \sqrt[3]{\frac{3V}{4\pi}}$

$\frac{dV}{dt} = -4\pi k\left(\frac{3V}{4\pi}\right)^{2/3}$, but all the numbers could be included in the constant

So... $\boxed{\frac{dV}{dt} = -kV^{2/3} \text{ for } k > 0}$

1.2 Solutions of Some Diff. Eq's

p. 15: 1, 3, 5, 8, 14, 17 (a,b), 18

1.) a.) $\frac{dy}{dt} = -y + 5 \rightarrow \int \frac{dy}{5-y} = \int dt$

$-\ln|5-y| = t + c \rightarrow 5 - y = Ce^{-t} \rightarrow y = 5 - Ce^{-t}$

If $y(0) = y_0$ then

$\boxed{y = 5 + (y_0 - 5)e^{-t}}$

b.) $\frac{dy}{dt} = -2y + 5 \rightarrow \int \frac{dy}{5-2y} = \int dt$

$-\frac{1}{2}\ln|5-2y| = t + c \rightarrow 5 - 2y = Ce^{-2t}$

$\rightarrow y = 5/2 + (y_0 - 5/2)e^{-2t}$

c.) $\frac{dy}{dt} = -2y + 10 \rightarrow \boxed{y = 5 + (y_0 - 5)e^{-2t}}$

3.) $\frac{dy}{dt} = -ay + b$

a.) $y = \frac{b}{a} \pm ce^{-at}$ b.) explore on calculator

c.) i) equilibrium is lower & approached rapidly
ii) equilibrium is higher
iii) equilibrium remains constant

5.) a.) $\frac{dy}{dt} = ay \rightarrow \frac{dy}{y} = a\,dt \rightarrow \ln|y| = at + c \rightarrow \boxed{y_1 = Ce^{at}}$

b.) $\frac{dy}{dt} = ay - b$ Use $y = Ce^{at} + k$, so $k = b/a$

c.) Same

8.) $\frac{dp}{dt} = rp \rightarrow p(t) = Ce^{rt}$

a.) $2C = Ce^{r(30)} \rightarrow \ln(2) = 30r \rightarrow r = \frac{\ln 2}{30} = .0231$

b.) $r = \frac{\ln 2}{N}$

14.) $\frac{1}{2} = e^{1620k} \rightarrow k = \frac{\ln(1/2)}{1620}$

$\frac{3}{4} = e^{\frac{\ln(1/2)}{1620}t} \rightarrow \ln(3/4) = \frac{\ln(1/2)}{1620} t$

$t = \frac{1620 \ln 3/4}{\ln 1/2} = \boxed{672.36 \text{ yrs.}}$

17.) $R\frac{dQ}{dt} + \frac{Q}{C} = V$

a.) $\frac{dQ}{dt} = \frac{V}{R} - \frac{Q}{CR}$ $\qquad \frac{b}{a} - ke^{-at}$

$Q(t) = VC - ke^{-t/CR}$

$Q(0) = VC - k = 0$

$k = VC$

$\boxed{Q(t) = VC(1 - e^{-t/CR})}$

b.) $Q_L = VC$ $\quad (\lim_{t\to\infty} Q(t) = VC)$

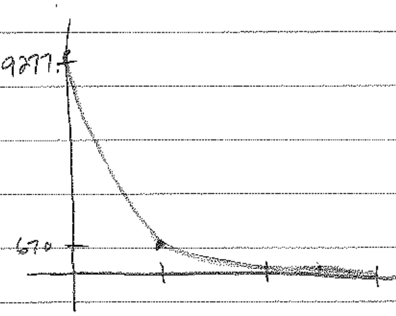
18.) a.) $\frac{dQ}{dt} = 300\left(.01 - \frac{Q}{1{,}000{,}000}\right) = 3\left(1 - \frac{Q}{10000}\right)$ $\qquad Q(0) = 0$

b.) $Q(t) = 10000 - ke^{-3t/10000} \rightarrow k = 10000$

$Q(t) = 10000 - 10000e^{-3t/10000}$

1 yr = 8760 hrs.

$Q(8760) = 9277.77$ grams

f.) 

c.) $\frac{dQ}{dt} = -\frac{3Q}{10000}$ $\qquad Q(0) = 9277.77$

d) $\ln|Q| = \frac{-3t}{10000} + k$ $\quad Q(t) = ke^{-3t/10000} \rightarrow \boxed{Q(t) = 9277.77e^{-3t/10000}}$

$Q(8760) = 670.07$ grams

e.) $Q(t) = 10$ when $t = 22775.97$ hrs.

or 2.6 years

1.3 Classifying Differential Equations

p. 24 : 1-6, 8, 12, 14, 15, 17, 19, 21, 22, 23, 26

1.) $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin t$ 2nd order ; linear

2.) $(1+y^2)\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = e^t$ 2nd order ; non-linear
$(1+y^2)$ not a function of t

3.) $\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1$ 4th order ; linear

4.) $\frac{dy}{dt} + ty^2 = 0$ 1st order ; non linear

5.) $\frac{d^2y}{dt^2} + \sin(t+y) = \sin t$ 2nd order ; non linear

6.) $\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (\cos^2 t)y = t^3$ 3rd order ; linear

8.) $y'' + 2y' - 3y = 0$ verify that $y_1(t) = e^{-3t}$ and $y_2(t) = e^t$ are solutions

$y_1'(t) = -3e^{-3t}$
$y_1''(t) = 9e^{-3t}$

$9e^{-3t} + (-6e^{-3t}) - 3e^{-3t} = 0$ ✓

$y_2'(t) = e^t$
$y_2''(t) = e^t$

$e^t + 2e^t - 3e^t = 0$ ✓

12.) $t^2y'' + 5ty' + 4y = 0$, $t > 0$

$y_1(t) = t^{-2}$
$y_1'(t) = -2t^{-3}$
$y_1''(t) = 6t^{-4}$

$t^2(6t^{-4}) + 5t(-2t^{-3}) + 4(t^{-2}) = 0$
$6t^{-2} - 10t^{-2} + 4t^{-2} = 0$ ✓

$y_2(t) = t^{-2}\ln t$
$y_2'(t) = -2t^{-3}\ln t + t^{-3}$
$y_2''(t) = 6t^{-4}\ln t - 2t^{-4} - 3t^{-4}$

$t^2(6t^{-4}\ln t - 5t^{-4}) + 5t(t^{-3} - 2t^{-3}\ln t) + 4(t^{-2}\ln t)$
$= \cancel{6t^{-2}\ln t} - \cancel{5t^{-2}} + \cancel{5t^{-2}} - \cancel{10t^{-2}\ln t} + \cancel{4t^{-2}\ln t}$
$= 0$ ✓

14.) $y' - 2ty = 1$ $y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}$

$y' = 2te^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}\cdot e^{-t^2} + \cancel{2te^{t^2}}$
$-2ty = -2te^{t^2}\int_0^t e^{-s^2}\,ds - \cancel{2te^{t^2}}$

$e^{t^2}\cdot e^{-t^2} = e^0 = 1$ ✓

15.) $y' + 2y = 0$ $\quad y = e^{rt}$ $\quad y' = re^{rt}$ $\quad y'' = r^2 e^{rt}$

$re^{rt} + 2e^{rt} = 0$

$e^{rt}(r+2) = 0 \rightarrow \boxed{r = -2}$

17.) $y'' + y' - 6y = 0$ $\quad (r+3)(r-2) = 0$

$r^2 e^{rt} + re^{rt} - 6e^{rt} = 0$ $\quad \boxed{r = -3 \text{ or } r = 2}$

$e^{rt}(r^2 + r - 6) = 0$

19.) $t^2 y'' + 4t y' + 2y = 0$ $\quad y = t^r$ $\quad y' = rt^{r-1}$ $\quad y'' = (r^2 - r)\, t^{r-2}$

$(r^2 - r)\cdot t^r + 4r \cdot t^r + 2t^r$ $\quad r^2 + 3r + 2$ $\quad \boxed{r = -2 \text{ or } r = -1}$

$t^r(r^2 - r + 4r + 2)$ $\quad (r+2)(r+1)$

21.) $u_{xx} + u_{yy} + u_{zz} = 0$ $\quad 2^{nd}$ order; linear

22.) $u_{xx} + u_{yy} + uu_x + uu_y + u = 0$ $\quad 2^{nd}$ order; nonlinear

23.) $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$ $\quad 4^{th}$ order; linear

26.) $\alpha^2 u_{xx} = u_t$ $\quad u_1(x,t) = e^{-\alpha^2 t}\sin x$ $\quad u_t = -\alpha^2 e^{-\alpha^2 t}\sin x$

$u_x = e^{-\alpha^2 t}\cos x$

$u_{xx} = -e^{-\alpha^2 t}\sin x$

$\alpha^2(-e^{-\alpha^2 t}\sin x) = -\alpha^2 e^{-\alpha^2 t}\sin x$ ✓

$u_2(x,t) = e^{-\alpha^2\lambda^2 t}\sin\lambda x$

$u_x = e^{-\alpha^2\lambda^2 t}\cdot\lambda\cos\lambda x$

$u_{xx} = -\lambda^2 e^{-\alpha^2\lambda^2 t}\sin\lambda x$

$u_t = -\alpha^2\lambda^2 e^{-\alpha^2\lambda^2 t}\sin\lambda x$

$\alpha^2(-\lambda^2 e^{-\alpha^2\lambda^2 t}\sin\lambda x) = -\alpha^2\lambda^2 e^{-\alpha^2\lambda^2 t}\sin\lambda x$ ✓